PONZER

Riemann's Surfaces of the Modular Functions

$u^4 - v^4 + 2uv(1 - u^2v^2) = 0$

$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$

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The Riemann's Surfaces

-of the-

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-By-

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Thesis for the Degree of Bachelor of Science in Mathematics and Physics in the

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Ernest W Ponzer

ENTITLED The Riemann's Surfaces of the Modular Functions $u^4-v^4+2uv(1-u^2v^2)=0$ and $u^6-v^6+5u^2v^2(u^2-v^2)+4uv(1-u^4v^4)=0$

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Bachelor of Science in Maths. & Physics

S W Shattuck

HEAD OF DEPARTMENT OF Mathematics

The solution of algebraic equations of different degrees has always been a fruitful field of research for mathematicians since the beginning of the science. We have at present solutions for the quadratic, cubic and bi-quadratic; and in the "Comptes rendus" for 1858 M. Hermite showed that the general equation of the fifth degree can be solved by means of elliptic functions. The same paper, with some added discussion, is also found in his "Sur la théorie des équations modulaires et la résolution de l'équation du cinquième degré," published in the following year. In his solution Hermite made use of a resolvent (modular) equation corresponding to the trigonometric solution of the cubic. In this solution of the cubic

the equation is first reduced to the normal form:

$$4x^3 - 3x - a = 0,$$

where a is to be replaced by certain trigonometric functions. Similarly, Jerrard has shown that the quintic can be reduced to the normal form:

$$x^5 - x - a = 0,$$

where in this case a must be replaced by certain elliptic transcendents.

The resolvent equation in the form in which Hermite used it is:

$$\partial(u, v) \equiv u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0,$$

where u and v are certain elliptic functions. In discussing the properties of this resolvent Hermite employed Galois' theory, especially the theorem that certain non-symmetric functions of the roots of this equation can be expressed rationally

in terms of the coefficients of the equation. All the theorems employed by Hermite are proved in C. Jordan's "Traité des substitutions et des équations algébriques," which appeared in 1870.

In an article in Schlömilch's Zeitschrift, Vol. 25, 1880, entitled "Über Hermite's Auflösung der Gleichung fünften Grades", Krey has discussed the above modular function at some length and has given proofs of many of its properties.

Starting then with the two functions given above, we may ask ourselves, what light the methods of ordinary function theory will throw on them. An elegant method of studying the function of a complex variable is that of spreading

the various branches of the function over the arbitrarily defined surface, due to Riemann, which surface consists of a number of parallel planes connected with one another, the sheets going over into each other without intersecting. We shall then consider the Riemann's Surfaces of the implicit functions:

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0$$

and

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0.$$

The method of studying the Riemann's Surface of an implicit function is the following: In the first place we cannot express either variable explicitly as a simple function of the other, hence we must devise some other method than that ordinarily used in getting the Riemann's surface of the function of a variable.

We arrange the function according to the descending powers of one of the variables, either of which may be considered as the independent one. The coefficients of this variable will be functions of the other variable. Another step is to find those values of the independent variable which will make the dependent variable equal to zero or infinite in value. We must next find the critical and branch points. Having then given:

(1) $\mathfrak{F}(\overset{n}{u}, \overset{m}{v}) = 0$, to obtain the critical points we form:

(2) $\dfrac{\partial \mathfrak{F}(\overset{n}{u}, \overset{m}{v})}{\partial \overset{n}{u}} = 0.$ where $\underline{n}$ and $\underline{m}$ represent the degree of the function in $\underline{u}$ and $\underline{v}$ respectively.

Eliminating $\underline{u}$ between (1) and (2) we get the resultant in $\underline{v}$, the solution

of which gives us the critical or branch points according as to whether we consider u as the dependent or independent variable respectively.

We shall first discuss thoroughly the former of the two given functions, and in the study of its Riemann's Surface methods will suggest themselves which will make the analogous work for the latter function much simpler and more elegant. We then have the function:

$$u^4 - v^4 + 2uv(1-u^2v^2) = 0,$$

and we obtain its partial derivative with respect to u. The latter will then be of the third degree in u, hence when we employ Sylvester's Dialytic Method of Elimination we shall have a seven row determinant in the functions of v, which are considered as coefficients of u.

We have then:

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0.$$

Or: $u^4 - v^4 + 2uv - 2u^3v^3 = 0$

Now forming $\dfrac{\partial F(\underline{u}, \underline{v})}{\partial \underline{u}}$ and arranging both according to the descending powers of $\underline{u}$ we have:

(1) $u^4 - 2u^3v^3 + 2uv - v^4 = 0$

(2) $4u^3 - 6u^2v^3 + 2v \qquad = 0.$

Multiplying $\underline{(1)}$ by $\underline{u}^2$ and $\underline{u}$ and the second by $\underline{u}^3$, $\underline{u}^2$ and $\underline{u}$ we have:

$$
\begin{array}{lllllllll}
u^6 & -2u^5v^3 & +\ 0 & +2u^3v & -u^2v^4 & & & & = 0 \\
 & u^5 & -2u^4v^3 & +\ 0 & +2u^2v & -uv^4 & & & = 0 \\
 & & u^4 & -2u^3v^3 & +\ 0 & +2uv & -v^4 & & = 0 \\
4u^6 & -6u^5v^3 & +\ 0 & +2u^3v & & & & & = 0 \\
 & 4u^5 & -6u^4v^3 & +\ 0 & +2u^2v & & & & = 0 \\
 & & 4u^4 & -6u^3v^3 & +\ 0 & +2uv & & & = 0 \\
 & & & 4u^3 & -6u^2v^3 & +\ 0 & +2v & & = 0
\end{array}
$$

We now form the determinant of the coefficients of $\underline{u}$ and have the following:

$$\begin{vmatrix} 1 & -2v^3 & 0 & 2v & -v^4 & 0 & 0 \\ 0 & 1 & -2v^3 & 0 & 2v & -v^4 & 0 \\ 0 & 0 & 1 & -2v^3 & 0 & 2v & -v^4 \\ 4 & -6v^3 & 0 & 2v & 0 & 0 & 0 \\ 0 & 4 & -6v^3 & 0 & 2v & 0 & 0 \\ 0 & 0 & 4 & -6v^3 & 0 & 2v & 0 \\ 0 & 0 & 0 & 4 & -6v^3 & 0 & 2v \end{vmatrix} = 0$$

Take out $\underline{v}^3$ and subtract four times the first row from the fourth row. The determinant then becomes:

$$v^3 \begin{vmatrix} 1 & -2v^3 & 0 & 2 & v^3 & 0 \\ 0 & 1 & -2v^3 & 0 & 2 & -v^3 \\ 2v^3 & 0 & -6v & 4v^3 & 0 & 0 \\ 4 & -6v^3 & 0 & 2 & 0 & 0 \\ 0 & 4 & -6v^3 & 0 & 2 & 0 \\ 0 & 0 & 4 & -6v^2 & 0 & 2 \end{vmatrix} = 0$$

Take out $\underline{2v}^4$; multiply last row by $\underline{v}^3$ and add to second row. The determinant becomes:

$$v^4 \begin{vmatrix} 1 & -2v^3 & 0 & 2 & -v^3 \\ 0 & 1 & 0 & -3v^5 & 2 \\ v^3 & 0 & -1 & 2v^3 & 0 \\ 2 & -3v^3 & 0 & 1 & 0 \\ 0 & 2 & -v^2 & 0 & 1 \end{vmatrix} = 0.$$

Now put third column as first; multiply third row by v^2 and subtract from the last row. The determinant then becomes:

$$v^4 \begin{vmatrix} 1 & 2v^3 & 2 & -v^3 \\ 0 & -1 & -3v^5 & 2 \\ 2 & 3v^3 & 1 & 0 \\ -v^5 & -2 & -2v^5 & 1 \end{vmatrix} = 0.$$

Subtract twice the first row from the third, and add first multiplied by v^5 to fourth. The determinant then becomes:

$$v^4 \begin{vmatrix} -1 & -3v^5 & 2 \\ -v^3 & -3 & 2v^3 \\ 2v^8-2 & 0 & 1-v^8 \end{vmatrix} = 0.$$

Take out 3, put second column as first and then subtract second row multiplied by v^5 from the first. We then have:

$$v^4 \begin{vmatrix} 1-v^8 & 2-2v^8 \\ 2-2v^8 & 1-v^8 \end{vmatrix} = 0$$

$$\therefore\ v^4\left[(1-v^8)^2 - (2-2v^8)^2\right] = 0$$

or $\quad v^4(1-2v^8+v^{16}-4+8v^8-4v^{16}) = 0$

This reduces to:

$$v^4(v^8-1)^2=0$$

$$\therefore v^4=0 \quad \text{and} \quad (v^8-1)=0$$

$$\therefore v=\sqrt[8]{1}.$$

This we may put in the form:

$$v=z^8=2k\pi+i\sin 2k\pi$$

$$\therefore z=\frac{2k}{8}\pi+i\sin\frac{2k}{8}\pi$$

Changing back to our variable $\underline{v}$ we have:

For:

$k=1$,	$v=e^{\frac{1}{4}\pi i}$	$k=5$,	$v=e^{\frac{5}{4}\pi i}$
$k=2$,	$v=e^{\frac{1}{2}\pi i}$	$k=6$,	$v=e^{\frac{3}{2}\pi i}$
$k=3$,	$v=e^{\frac{3}{4}\pi i}$	$k=7$,	$v=e^{\frac{7}{4}\pi i}$
$k=4$,	$v=e^{\pi i}$	$k=8$,	$v=1$.

$\underline{v=0}$ and $\underline{v=\infty}$ are both branch points of order $\underline{2}$.

To show that $\underline{v=\infty}$ is a branch point we first make the transformations:

$$\bar{v}=\frac{1}{v} \therefore v=\frac{1}{\bar{v}} \quad \text{and} \quad \bar{u}=\frac{1}{u} \therefore u=\frac{1}{\bar{u}}.$$

This transformation brings the infinite region of the plane about the origin where we can more readily handle the function.

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Making then these transformations in our original function (u is infinite when v is infinite) we have the following:

$$\frac{1}{\bar{u}^4} - \frac{1}{\bar{v}^4} + 2\frac{1}{\bar{u}}\cdot\frac{1}{\bar{v}}\left(1 - \frac{1}{\bar{u}^2}\cdot\frac{1}{\bar{v}^2}\right) = 0$$

or $\quad \bar{v}^4 - \bar{u}^4 + 2\bar{u}^3\bar{v}^3 - 2\bar{u}\bar{v} = 0 \quad$ Mult. by -1.

$$\bar{u}^4 - \bar{v}^4 + 2\bar{u}\bar{v}\left(1 - \bar{u}^2\bar{v}^2\right) = 0.$$

We then treat this function in $\bar{u}$ and $\bar{v}$ exactly as the original function in u and v, and we see that $\bar{u}$ and $\bar{v}$ are roots of the above equation.

Since it is immaterial whether we consider u or v as the independent variable we may then with equal right regard the ten resulting values of v either as critical or branch points. We choose the latter, hence we must then find the critical points corresponding to the different branch points. This is

done by simply substituting the various values of v given above in the function:

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0$$

and then solving for the value of u. This method gives us four values of u in each case, three of which will be equal, representing the critical point, while the fourth is the ordinary point corresponding to the particular value of v employed.

The details of this work involve only algebraic operations and the fact that:

$$e^{2n\pi i} = -e^{(2n+1)\pi i} = e^{(2n+2)\pi i}$$; hence we give only the results obtained.

In solving for the critical and branch points we also find that the <u>orders</u> are the same. We also find the ordinary points corresponding to the various critical points. For the sake of convenience we

arrange these values in tabular form, and below give a graphical representation of the relation between the values of $\underline{u}$ and $\underline{v}$.

Critical and Branch Points.

Crit. Point	Order	Branch Pt.	Times	Ordinary Pt.
1	2	-1	3	1
$e^{\frac{1}{4}\pi i}$	2	$-e^{\frac{3}{4}\pi i}$	3	$e^{\frac{3}{4}\pi i}$
$e^{\frac{1}{2}\pi i}$	2	$e^{\frac{1}{2}\pi i}$	3	$-e^{\frac{1}{2}\pi i}$
$e^{\frac{3}{4}\pi i}$	2	$-e^{\frac{1}{4}\pi i}$	3	$e^{\frac{1}{4}\pi i}$
$e^{\pi i}$	2	1	3	-1
$-e^{\frac{1}{4}\pi i}$	2	$e^{\frac{3}{4}\pi i}$	3	$-e^{\frac{3}{4}\pi i}$
$-e^{\frac{1}{2}\pi i}$	2	$-e^{\frac{1}{2}\pi i}$	3	$e^{\frac{1}{2}\pi i}$
$-e^{\frac{3}{4}\pi i}$	2	$e^{\frac{1}{4}\pi i}$	3	$-e^{\frac{1}{4}\pi i}$
0	2	0	3	0
∞	2	∞	3	∞

$\underline{u}$ $\qquad$ O = Ordinary $\qquad$ $\underline{v}$

B = Branch.

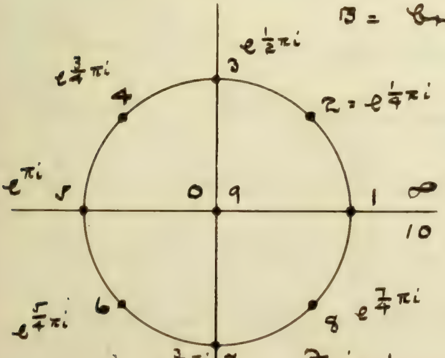

Fig 1.

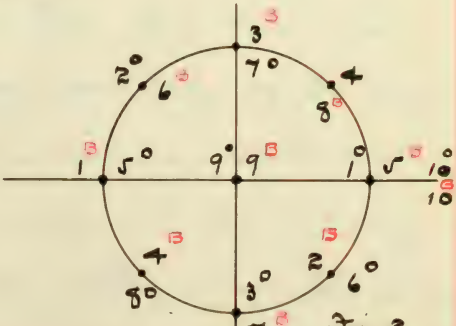

Fig 2.

We can now see that there are four sheets to the Riemann's Surface of the function under consideration, and that they are connected at the branch points given above. We also know, since the critical points are of the second order, that only three sheets hang together at any one point. Then there must be a smooth sheet at each point, as is also shown by the fact that in each case we find a corresponding ordinary point. Now the question arises: "How are the sheets connected?" We may start in by numbering the sheets arbitrarily, which we shall do, as will be shown later, but we can do this for only one point. The ordinary methods of Klein's Geometrical Function Theory will not answer here, as we cannot by solving algebraically express either variable as a

simple function of the others; and hence we shall adopt the following method:

At each of the branch points we develop $\underline{v}$ as a power series in $\underline{u}$ for each of the four branches of the function at that point. This will then give us four expansions for each point. This is Cauchy's method for getting criteria for the order in which the sheets should be connected. Having then obtained the four developments about a point, or in its vicinity, we may then say arbitrarily which of the three sheets that hang together shall go over into each other at that point. Then by computing the values of $\underline{v}$ for a point within the common region of convergence of two sets of developments we shall have sufficient data to show us how the sheets should be connected at every point. The same computations

will also show which of the sheets remains smooth.

The method being outlined, let us now consider the details of this somewhat cumbersome method.

We have given the function:

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0,$$

and we wish to express $\underline{v}$ in terms of $\underline{u}$ and also $\underline{u}$ in terms of $\underline{v}$. We can express either variable as the product of a power series in the other variable. For instance:

$$\text{Let } v = \eta u$$

$$\text{where } \eta = c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4 + \cdots$$

By substituting these values in our original expression and comparing coefficients we get the series represented by η, which was at first assumed arbitrarily, as above.

I employed the methods as given by Stolz in his "Allgemeine Arithmetik", Vol. I, p. 296, and as given in Chrystal's Algebra,

Vol 2, p. 349.

We shall first proceed with the developments of $\underline{v}$ corresponding to the critical point $\underline{u} = \underline{0}$.

We have the function:

(1) $u^4 - v^4 + 2uv(1 - u^2v^2) = 0$ in which we put:

(2) $v = \eta u$ where we define η as:

(3) $\eta = \tau_1 u + \tau_2 u^2 + \tau_3 u^3 + \tau_4 u^4 + \text{----}$

We then have:

(4) $u^4 - \eta^4 u^4 + 2\eta u^2 - 2\eta^3 u^6 = 0$

or $\eta^4 u^4 + 2\eta^3 u^6 - 2\eta u^2 - u^4$ which we put into the form:

(5) $-u^2(2\eta + u^2 - \eta^4 u^2 - 2\eta^3 u^4) = 0.$

Since $u \neq 0$ then we must have:

(6) $2\eta = -u^2 + \eta^4 u^2 + 2\eta^3 u^4$

Now substituting the value of η we have:

$$2(\tau_1 u + \tau_2 u^2 + \tau_3 u^3 + \tau_4 u^4 + \text{----})$$
$$= -u^2 + u^2(\tau_1 u + \tau_2 u^2 + \text{---})^4 + 2u^4(\tau_1 u + \text{---})^3$$

as the two sides, or members of the equation, in which we compare coefficients.

At this stage it will be seen that we must raise an infinite series to the third and fourth powers; later we shall need the second, fifth and sixth powers of the same series. Before proceeding further we shall then give the expansions of the following series:

$$(c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4 + \cdots)^n$$

$$\text{where} \quad n = 1, 2, 3, 4, 5, 6.$$

We shall gather together the coefficients of the various powers of u, designating each coefficient by a proper symbol, which symbols shall be the letters of the alphabet starting with:

$$J_1, J_2, J_3, J_4 \cdots\cdots$$

and these shall represent the coefficients of the powers of u in the expansions of the series to the different powers.

Symbolically we then have as our expansions the following:

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$$(c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4 + \text{------})^2$$
$$= J_1 u^2 + J_2 u^3 + J_3 u^4 + J_4 u^5 + \text{-----}$$

where: $J_1 = {}^2\Sigma\, c_p c_q$ where the number outside of the sign of summation represents the sum of the subscripts and $\underline{p}$ and $\underline{q}$

$J_2 = {}^3\Sigma\, c_p c_q$

$J_3 = {}^4\Sigma\, c_p c_q$

$J_4 = {}^5\Sigma\, c_p c_q$

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any two positive integers whose sum is equal to the number outside.

Similarly we then have for:

$(c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4 + \text{-----})^3$, calling the coefficients of the powers of $\underline{u}$ $\underline{K}$'s the following:

$K_1 = {}^3\Sigma\, c_p c_q c_r$ where the same law holds for the sum of the subscripts as in the above.

$K_2 = {}^4\Sigma\, c_p c_q c_r$

$K_3 = {}^5\Sigma\, c_p c_q c_r$

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I call the coefficients of the powers of $\underline{u}$

in the expansion of the series to the fourth power:

$$L_1, L_2, L_3, L_4 \text{------};$$

those for the fifth power of the series:

$$M_1, M_2, M_3, M_4 \text{-------};$$

and those for the sixth power:

$$N_1, N_2, N_3, N_4 \text{--------}.$$

I shall use this nomenclature in my developments, the expanded coefficients being found in several treatises on higher algebra.

We can now substitute for the expansions of η to the different powers more readily; and then by comparing coefficients on both sides we find:

Since:

$$2(c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4 + \text{------})$$

$$= -u^2 + 2u^4(\kappa_1 u^3 + \kappa_2 u^4 + \kappa_3 u^5 + \text{----})$$

$$+ u^2(L_1 u^4 + L_2 u^5 + L_3 u^6 + \text{------})$$

that we have the following comparisons:

$$2c_1 = 0 \qquad \therefore \qquad c_1 = 0$$

$$2c_2 = -1 \qquad c_2 = -\tfrac{1}{2}$$

$$2c_3 = 0 \qquad c_3 = 0$$

$$2c_4 = 0 \qquad c_4 = 0$$

$$2c_5 = 0 \qquad c_5 = 0$$

$$2c_6 = L_1 \qquad c_6 = \frac{L_1}{2}$$

$$2c_7 = 2\pi_1 + L_2 \qquad c_7 = \pi_1 + \frac{L_2}{2}$$

$$2c_8 = 2\pi_2 + L_3 \qquad c_8 = \pi_2 + \frac{L_3}{2}$$

$$2c_9 = 2\pi_3 + L_4 \qquad c_9 = \pi_3 + \frac{L_4}{2}$$

$$2c_{10} = 2\pi_4 + L_5 \qquad c_{10} = \pi_4 + \frac{L_5}{2}$$

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$$2c_{18} = 2\pi_{12} + L_{13} \qquad \therefore \qquad c_{18} = \pi_{12} + \frac{L_{13}}{2}$$

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$$2c_{26} = 2\pi_{20} + L_{21} \qquad c_{26} = \pi_{20} + \frac{L_{21}}{2}$$

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It will be seen that we can then determine every $\underline{c}$, and when we have determined them as far as we wish we may write out the series represented by η.

Without giving further computations we

have as the expansion for η:

$$\eta = -\frac{1}{2}u^2 - \frac{3}{32}u^{10} - \frac{3}{64}u^{18} - \frac{123}{4096}u^{26} - \cdots\cdots .$$

This gives us one branch at $\underline{u} = \underline{0}$. To find the other branches at $\underline{u} = \underline{0}$

Let $u = \eta v$ and substitute this value in our original equation.

Then:

$$\eta^4 v^4 - v^4 + 2\eta v^2 - 2\eta^3 v^6 = 0$$

$$\text{or } 2\eta = v^2 - \eta^4 v^2 + 2\eta^3 v^4.$$

Here let:

$$\eta = (c_1 v + c_2 v^2 + c_3 v^3 + c_4 v^4 + \cdots\cdots).$$

Expand and equate coefficients, and by a process in every similar to that above we obtain: Since $\underline{u} = \underline{\eta v}$.

$$u = \frac{1}{2}v^3 + \frac{3}{32}v^{11} + \frac{3}{64}v^{19} + \frac{123}{4096}u^{27} + \frac{177}{8192}u^{35} + \cdots$$

We now have $\underline{u}$ as a function of $\underline{v}$; but as we have chosen $\underline{v}$ to represent our dependent variable, or our branch point when $\underline{u}$ represents the critical point then we know that as each critical point is

of the second order, the angles there are trebled at the branch point. We want then an expansion where v is a function of u; so, having u as a function of v we revert the series by assuming:

$$v = c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4 + \cdots\cdots$$

and then compare coefficients. This will give us the three expansions at the branch point if we substitute successively:

$$v_1 = c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4 + \cdots\cdots$$

$$v_2 = c_1 (\omega \xi) + c_2 (\omega \xi)^2 + c_3 (\omega \xi)^3 + \cdots\cdots$$

$$v_3 = c_1 (\omega^2 \xi) + c_2 (\omega^2 \xi)^2 + c_3 (\omega^2 \xi)^3 + \cdots\cdots.$$

We have the equation:

$$u = \frac{1}{2} v^3 + \frac{3}{32} v^{11} + \frac{3}{64} v^{19} + \frac{123}{4096} v^{27} + \frac{177}{8192} v^{35} + \cdots$$

Changing the form we have:

$$2u = v^3 \left(1 + \frac{3}{16} v^8 + \frac{3}{32} v^{16} + \frac{123}{2048} v^{24} + \frac{177}{4096} v^{32} + \cdots\right)$$

Extracting the cube root of both sides we have:

$$(2u)^{\frac{1}{3}} = \sqrt[3]{2}\,(\omega \xi) = v + \frac{1}{16} v^9 + \frac{7}{256} v^{17} + \frac{203}{12288} v^{25} + \frac{2251}{98304} v^{33} + \cdots\cdots$$

If we now substitute the values of v_2 and v_3 and reduce similarly, compare and equate coefficients, thus obtaining the c's we get the following three expansions of v about the point $u = 0$:

$$\text{I.}\quad v_1 = \sqrt[3]{2}\, u^{\frac{1}{3}} - \frac{1}{2} u^{\frac{9}{3}} + \frac{1}{4}\sqrt[3]{4}\, u^{\frac{17}{3}} - \frac{1}{6}\sqrt[3]{2}\, u^{\frac{25}{3}} + \frac{1}{32} u^{\frac{33}{3}} + \cdots$$

$$\text{II.}\quad v_2 = \sqrt[3]{2}\,\omega u^{\frac{1}{3}} - \frac{1}{2} u^{\frac{9}{3}} + \frac{1}{4}\sqrt[3]{4}\,\omega^2 u^{\frac{17}{3}} - \frac{1}{6}\sqrt[3]{2}\,\omega u^{\frac{25}{3}} + \frac{1}{32} u^{\frac{33}{3}} \cdots$$

$$\text{III.}\quad v_3 = \sqrt[3]{2}\,\omega^2 u^{\frac{1}{3}} - \frac{1}{2} u^{\frac{9}{3}} + \frac{1}{4}\sqrt[3]{4}\,\omega u^{\frac{17}{3}} - \frac{1}{6}\sqrt[3]{2}\,\omega^2 u^{\frac{25}{3}} + \frac{1}{32} u^{\frac{33}{3}} \cdots$$

We now have the developments for the four branches of the function at the point $v = 0$, corresponding to the critical point $u = 0$. But we must have similar developments for each of the branch points. The following method of treating the point:

$$u = 1$$

and of obtaining the developments for the four branches of the function — three at the branch point corresponding to:

$$u = 1 \text{ which is:}$$

$$v = -1$$ and one corresponding

to $v = 1$, the ordinary point,—can be applied to all the other critical points.

By referring to the table of critical and branch points it is seen that for:

$u = 1$. Ord. Pt. is $v = 1$., Br. Pt is $v = -1$.

To develop the function about $v = 1$ we

$$\text{Let } \bar{u} = u - 1 \quad \therefore u = \bar{u} + 1$$

$$\text{Then } \bar{v} = v - 1 \quad \therefore v = \bar{v} + 1.$$

Substituting these values for u and v in our original function we have:

$$(\bar{u}+1)^4 - (\bar{v}+1)^4 + 2(\bar{u}+1)(\bar{v}+1) - 2(\bar{u}+1)^3(\bar{v}+1)^3 = 0$$

When the binomials are expanded and like terms grouped we have:

$$8\bar{v} = \bar{u}^4 - \bar{v}^4 + 2\bar{u}^3 - 16\bar{u}\bar{v} - 18\bar{u}^2\bar{v} - 6\bar{u}^3\bar{v}$$
$$- 12\bar{v}^2 - 18\bar{u}\bar{v}^2 - 18\bar{u}^2v^2 - 6\bar{u}^3v^2 - 6\bar{v}^3 - 6\bar{u}\bar{v}^3$$
$$- 6\bar{u}^2\bar{v}^3 - 2\bar{u}^3v^3 = 0.$$

Assuming here that:

$$v = c_1u + c_2u^2 + c_3u^3 + c_4u^4 + \cdots$$

we see that we can get v in terms of u

by equating coefficients and solving for the $\underline{c's}$. This gives us the expansion at the ordinary point.

Similarly to get the expansions at the branch points we must first make the transformations: (See Table)

$$\bar{u} = u - 1 \qquad \therefore\ u = \bar{u} + 1$$
$$\bar{v} = v + 1 \qquad \therefore\ v = \bar{v} - 1.$$

Again substituting these values and reducing we have:

$$8\bar{u} = -12\bar{u}^2 + 16\bar{u}\bar{v} - 6\bar{u}^3 + 18\bar{u}^2\bar{v} - 18\bar{u}\bar{v}^2$$
$$-2\bar{v}^3 - \bar{u}^4 + 6\bar{u}^3\bar{v} - 18\bar{u}^2\bar{v}^2 + 6\bar{u}\bar{v}^3$$
$$+\bar{v}^4 - 6\bar{u}^3\bar{v}^2 + 6\bar{u}^2\bar{v}^3 + 2\bar{u}^3\bar{v}^3.$$

In this equation after finding $\underline{u}$ as a function of $\underline{v}$ we revert and express $\underline{v}$ as a function of $\underline{u}$. After reversion we shall have four developments of the function about $\underline{u} = \underline{1}$.

To obtain the developments about the other critical points it is evident that

we need only make the transformations:

$$\bar{u} = u - e^{\frac{\pi}{4}\pi i} \quad \therefore\ u = \bar{u} + e^{\frac{\pi}{4}\pi i}$$

and $$\bar{v} = v \mp e^{\frac{f}{4}\pi i} \quad \therefore\ v = \bar{v} \pm e^{\frac{f}{4}\pi i}$$

the transformations required for v being suggested by the table of critical and branch points. The work, though long and tedious, is straight forward, hence we give only the results of the computations.

Arranging our expansions at the various critical points we have the table of developments given below. These developments are all power series in u; hence they are all convergent within some definite region; and according to Cauchy we know that the region of convergence is up to the next critical point, but not including it.

We give the expansion for the ordinary point first and then one of the three at the branch points, choosing for illustration

the one corresponding to $\underline{v_2}$ as we have designated it heretofore. $\underline{v_1}$ is obtained from $\underline{v_2}$ by simply dropping the $\underline{\omega's}$; while $\underline{v_3}$ is obtained by substituting $\underline{\omega^2}$ for $\underline{\omega}$ in the expansion for $\underline{v_2}$ and $\underline{\omega}$ for $\underline{\omega^2}$.

Throughout: $\quad \omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$

and $\quad \omega^2 = -\frac{1}{2} - \frac{1}{2}\sqrt{-3}$

Table of Developments:

About $\underline{u} = \underline{0}$.

$$v = -\frac{1}{2}u^3 - \frac{3}{32}u^{11} - \frac{3}{64}u^{19} - \frac{123}{4096}u^{27} - \cdots\cdots$$

$$v = \sqrt[3]{2}\,\omega\, u^{\frac{1}{3}} - \frac{1}{2}u^{\frac{9}{3}} + \frac{1}{4}\sqrt[3]{4}\,\omega^2 u^{\frac{17}{3}} - \frac{1}{6}\sqrt[3]{2}\,\omega\, u^{\frac{25}{3}} + \frac{1}{32}u^{\frac{33}{3}} \cdots$$

About $\underline{u} = \underline{1}$.

$$v - 1 = \frac{1}{4}(u-1)^3 - \frac{3}{8}(u-1)^4 + \frac{3}{16}(u-1)^5 + \frac{3}{16}(u-1)^6$$

$$- \frac{3}{8}(u-1)^7 + \frac{15}{128}(u-1)^8 + \frac{3}{8}(u-1)^9 - \cdots\cdots$$

$$v + 1 = -\sqrt[3]{4}.\,\omega(u-1)^{\frac{1}{3}} - \sqrt[3]{2}.\,\omega^2(u-1)^{\frac{2}{3}} - 2(u-1)^{\frac{3}{3}}$$

$$- \frac{4}{3}\sqrt[3]{4}.\,\omega(u-1)^{\frac{4}{3}} - \frac{5}{3}.\sqrt[3]{2}.\,\omega^2(u-1)^{\frac{5}{3}} - 2(u-1)^{\frac{6}{3}} \cdots$$

About $u = e^{\frac{1}{4}\pi i}$

$$v - e^{\frac{3}{4}\pi i} = \frac{1}{4}\left(u - e^{\frac{1}{4}\pi i}\right)^3 + \frac{3}{8} e^{\frac{3}{4}\pi i}\left(u - e^{\frac{1}{4}\pi i}\right)^4$$

$$- \frac{3}{16} e^{\frac{1}{2}\pi i}\left(u - e^{\frac{1}{4}\pi i}\right)^5 - \frac{3}{16} e^{\frac{3}{4}\pi i}\left(u - e^{\frac{1}{4}\pi i}\right)^6$$

$$+ \frac{3}{8}\left(u - e^{\frac{1}{4}\pi i}\right)^7 + \frac{15}{128} e^{\frac{3}{4}\pi i}\left(u - e^{\frac{1}{4}\pi i}\right)^8 + \frac{3}{8} e^{\frac{1}{2}\pi i}\left(u - e^{\frac{1}{4}\pi i}\right)^9 \ldots$$

$$v + e^{\frac{3}{4}\pi i} = -\sqrt[3]{4}.\,\omega\left(u - e^{\frac{1}{4}\pi i}\right)^{\frac{1}{3}} + \sqrt[3]{2}.\,e^{\frac{1}{4}\pi i}.\,\omega^2\left(u - e^{\frac{1}{4}\pi i}\right)^{\frac{2}{3}} - 2 e^{\frac{1}{2}\pi i}\left(u - e^{\frac{1}{4}\pi i}\right)^{\frac{3}{3}}$$

$$+ \frac{4}{3}.\sqrt[3]{4}\,e^{\frac{3}{4}\pi i}.\,\omega\left(u - e^{\frac{1}{4}\pi i}\right)^{\frac{4}{3}} + \frac{5}{3}.\sqrt[3]{2}.\,\omega^2\left(u - e^{\frac{1}{4}\pi i}\right)^{\frac{5}{3}} - 2 e^{\frac{1}{4}\pi i}\left(u - e^{\frac{1}{4}\pi i}\right) \ldots$$

About $u = e^{\frac{1}{2}\pi i}$

$$v - e^{\frac{3}{2}\pi i} = \frac{1}{4}\left(u - e^{\frac{1}{2}\pi i}\right)^3 + \frac{3}{8} e^{\frac{1}{2}\pi i}\left(u - e^{\frac{1}{2}\pi i}\right)^4 - \frac{3}{16}.\,e^{\frac{1}{2}\pi i}\left(u - e^{\frac{1}{2}\pi i}\right)^6$$

$$- \frac{3}{8}\left(u - e^{\frac{1}{2}\pi i}\right)^7 - \frac{15}{128}.\,e^{\frac{1}{2}\pi i}\left(u - e^{\frac{1}{2}\pi i}\right)^8 - \frac{3}{8}\left(u - e^{\frac{1}{2}\pi i}\right)^9 \ldots$$

$$v + e^{\frac{3}{2}\pi i} = -\sqrt[3]{4}.\,\omega\left(u - e^{\frac{1}{2}\pi i}\right)^{\frac{1}{3}} - \sqrt[3]{2}\,e^{\frac{1}{2}\pi i}.\,\omega^2\left(u - e^{\frac{1}{2}\pi i}\right)^{\frac{2}{3}} + 2\left(u - e^{\frac{1}{2}\pi i}\right)^{\frac{3}{3}}$$

$$+ \frac{4}{3}.\sqrt[3]{4}.\,\omega\left(u - e^{\frac{1}{2}\pi i}\right)^{\frac{4}{3}} - \frac{5}{3}.\sqrt[3]{2}.\,\omega^2\left(u - e^{\frac{1}{2}\pi i}\right)^{\frac{5}{3}} - 2\,e^{\frac{1}{2}\pi i}\left(u - e^{\frac{1}{2}\pi i}\right)^{\frac{6}{3}} \ldots$$

About $u = e^{\frac{3}{4}\pi i}$

$$v - e^{\frac{1}{4}\pi i} = \frac{1}{4}\left(u - e^{\frac{3}{4}\pi i}\right)^3 + \frac{3}{8}.\,e^{\frac{1}{4}\pi i}\left(u - e^{\frac{3}{4}\pi i}\right)^4 + \frac{3}{16}.\,e^{\frac{1}{2}\pi i}\left(u - e^{\frac{3}{4}\pi i}\right)^5$$

$$- \frac{3}{16} e^{\frac{3}{4}\pi i}\left(u - e^{\frac{3}{4}\pi i}\right)^6 + \frac{3}{8}\left(u - e^{\frac{3}{4}\pi i}\right)^7 + \frac{15}{128}.\,e^{\frac{1}{4}\pi i}\left(u - e^{\frac{3}{4}\pi i}\right)^8$$

$$+ \frac{3}{8}.\,e^{\frac{1}{2}\pi i}\left(u - e^{\frac{3}{4}\pi i}\right)^9 \ldots\ldots$$

$$v + e^{\frac{1}{4}\pi i} = -\sqrt[3]{4}.\,\omega\left(u - e^{\frac{3}{4}\pi i}\right)^{\frac{1}{3}} + \sqrt[3]{2}.\,e^{\frac{3}{4}\pi i}.\,\omega^2\left(u - e^{\frac{3}{4}\pi i}\right)^{\frac{2}{3}} + 2 e^{\frac{1}{2}\pi i}\left(u - e^{\frac{3}{4}\pi i}\right)^{\frac{3}{3}}$$

$$+ \frac{4}{3}.\sqrt[3]{4}.\,\omega.\,e^{\frac{1}{4}\pi i}\left(u - e^{\frac{3}{4}\pi i}\right)^{\frac{4}{3}} + \frac{5}{3}.\sqrt[3]{2}.\,\omega^2\left(u - e^{\frac{3}{4}\pi i}\right)^{\frac{5}{3}} - 2 e^{\frac{3}{4}\pi i}\left(u - e^{\frac{3}{4}\pi i}\right)^{\frac{6}{3}} \ldots$$

About $u = e^{\pi i}$.

$$v - e^{\pi i} = \frac{1}{4}(u - e^{\pi i})^3 + \frac{3}{8}(u - e^{\pi i})^4 + \frac{3}{16}(u - e^{\pi i})^5 - \frac{3}{16}(u - e^{\pi i})^6$$

$$+ \frac{3}{8}(u - e^{\pi i})^7 - \frac{15}{128}(u - e^{\pi i})^8 + \frac{3}{8}(u - e^{\pi i})^9 - - - - - - -$$

$$v + e^{\pi i} = -\sqrt[3]{4}.\,\omega(u - e^{\pi i})^{\frac{1}{3}} + \sqrt[3]{2}.\,\omega^2(u - e^{\pi i})^{\frac{2}{3}} - 2(u - e^{\pi i})^{\frac{3}{3}}$$

$$+ \frac{4}{3}.\sqrt[3]{4}.\,\omega(u - e^{\pi i})^{\frac{4}{3}} - \frac{5}{3}.\sqrt[3]{2}.\,\omega^2(u - e^{\pi i})^{\frac{5}{3}} + 2(u - e^{\pi i})^{\frac{6}{3}} \ldots$$

About $u = e^{\frac{5}{4}\pi i}$.

$$v - e^{\frac{7}{4}\pi i} = \frac{1}{4}(u - e^{\frac{5}{4}\pi i})^3 - \frac{3}{8}.e^{\frac{3}{4}\pi i}(u - e^{\frac{5}{4}\pi i})^4 - \frac{3}{16}.e^{\frac{1}{2}\pi i}(u - e^{\frac{5}{4}\pi i})^5$$

$$+ \frac{3}{16}.e^{\frac{1}{4}\pi i}(u - e^{\frac{5}{4}\pi i})^6 - \frac{3}{8}(u - e^{\frac{5}{4}\pi i})^7 - \frac{15}{128}.e^{\frac{3}{4}\pi i}(u - e^{\frac{5}{4}\pi i})^8$$

$$+ \frac{3}{8}e^{\frac{1}{2}\pi i}(u - e^{\frac{5}{4}\pi i})^9 - - - - - - - - -$$

$$v + e^{\frac{7}{4}\pi i} = -\sqrt[3]{4}.\,\omega(u - e^{\frac{5}{4}\pi i})^{\frac{1}{3}} - \sqrt[3]{2}.e^{\frac{1}{4}\pi i}\,\omega^2(u - e^{\frac{5}{4}\pi i})^{\frac{2}{3}} + 2\,e^{\frac{1}{2}\pi i}(u - e^{\frac{5}{4}\pi i})^{\frac{3}{3}}$$

$$-\frac{4}{3}.\sqrt[3]{4}.\,e^{\frac{3}{4}\pi i}\,\omega(u - e^{\frac{5}{4}\pi i})^{\frac{4}{3}} + \frac{5}{3}.\sqrt[3]{2}.\,\omega^2(u - e^{\frac{5}{4}\pi i})^{\frac{5}{3}} + 2e^{\frac{1}{2}\pi i}(u - e^{\frac{5}{4}\pi i})^{\frac{6}{3}} \ldots$$

About $u = e^{\frac{3}{2}\pi i}$.

$$v - e^{\frac{1}{2}\pi i} = \frac{1}{4}(u - e^{\frac{3}{2}\pi i})^3 - \frac{3}{8}e^{\frac{1}{2}\pi i}(u - e^{\frac{3}{2}\pi i})^4 - \frac{3}{16}(u - e^{\frac{3}{2}\pi i})^5$$

$$-\frac{3}{16}e^{\frac{1}{2}\pi i}(u - e^{\frac{3}{2}\pi i})^6 - \frac{3}{8}(u - e^{\frac{3}{2}\pi i})^7 + \frac{15}{128}e^{\frac{1}{2}\pi i}(u - e^{\frac{3}{2}\pi i})^8$$

$$-\frac{3}{8}(u - e^{\frac{3}{2}\pi i})^9 - - - - - - - - -$$

$$v + e^{\frac{1}{2}\pi i} = -\sqrt[3]{4}.\,\omega(u - e^{\frac{3}{2}\pi i})^{\frac{1}{3}} + \sqrt[3]{2}.e^{\frac{1}{2}\pi i}\,\omega^2(u - e^{\frac{3}{2}\pi i})^{\frac{2}{3}} + 2(u - e^{\frac{3}{2}\pi i})^{\frac{3}{3}}$$

$$-\frac{4}{3}.\sqrt[3]{4}.e^{\frac{1}{2}\pi i}\,\omega(u - e^{\frac{3}{2}\pi i})^{\frac{4}{3}} - \frac{5}{3}\sqrt[3]{2}.\,\omega^2(u - e^{\frac{3}{2}\pi i})^{\frac{5}{3}} + 2e^{\frac{1}{2}\pi i}(u - e^{\frac{3}{2}\pi i})^{\frac{6}{3}} \ldots$$

About $u = e^{\frac{7}{4}\pi i}$

$$v - e^{\frac{5}{4}\pi i} = \frac{1}{4}\left(u - e^{\frac{7}{4}\pi i}\right)^3 - \frac{3}{8}\cdot e^{\frac{1}{4}\pi i}\left(u - e^{\frac{7}{4}\pi i}\right)^4 + \frac{3}{16} e^{\frac{1}{2}\pi i}\left(u - e^{\frac{7}{4}\pi i}\right)^5$$

$$+ \frac{3}{16} e^{\frac{3}{4}\pi i}\left(u - e^{\frac{7}{4}\pi i}\right)^6 + \frac{3}{8}\left(u - e^{\frac{7}{4}\pi i}\right)^7 - \frac{15}{128}\cdot e^{\frac{1}{4}\pi i}\left(u - e^{\frac{7}{4}\pi i}\right)^8$$

$$- \frac{3}{8}\cdot e^{\frac{1}{2}\pi i}\left(u - e^{\frac{7}{4}\pi i}\right)^9 - - - - - - - - - - - -$$

$$v + e^{\frac{5}{4}\pi i} = -\sqrt[3]{4}\cdot\omega\left(u - e^{\frac{7}{4}\pi i}\right)^{\frac{1}{3}} - \sqrt[3]{2}\, e^{\frac{3}{4}\pi i}\cdot\omega^2\left(u - e^{\frac{7}{4}\pi i}\right)^{\frac{2}{3}}$$

$$+ 2 e^{\frac{1}{2}\pi i}\left(u - e^{\frac{7}{4}\pi i}\right)^{\frac{3}{3}} - \frac{4}{3}\sqrt[3]{4}\cdot e^{\frac{1}{4}\pi i}\cdot\omega\left(u - e^{\frac{7}{4}\pi i}\right)^{\frac{4}{3}}$$

$$+ \frac{5}{3}\cdot\sqrt[3]{2}\cdot\omega^2\left(u - e^{\frac{7}{4}\pi i}\right)^{\frac{5}{3}} + 2 e^{\frac{3}{4}\pi i}\left(u - e^{\frac{7}{4}\pi i}\right)^{\frac{6}{3}} \cdots$$

The next step in the solution of our problem – that of connecting the sheets of the Riemann's Surface is to substitute some particular value for $\underline{u}$, which value shall lie within the regions of convergence for the function when developed about either $\underline{u} = \underline{0}$ or $\underline{u} = \underline{1}$. To facilitate operations I first chose a point at a distance of $\underline{0.8}$ from the origin in the $\underline{u}$ plane and on the axis

of reals. This point was then at a distance of 0.2 from the point $u = 1$. It is seen that this point is within the common region of convergence for the functions about $u = 0$ and also about $u = 1$.

The developments, according to Cauchy, about the point $u = 0$ are convergent within the circle about zero - full line; while the developments about $u = 1$ are convergent within the dotted circle - up to next critical point.

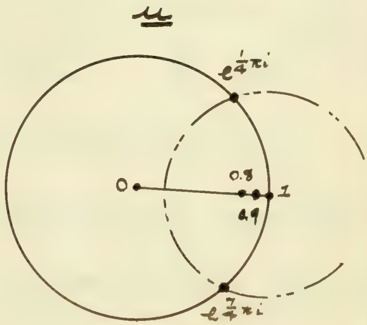


Fig. 3.

On substituting 0.8 for u in my developments I soon found that this value of u would not make my series converge rapidly enough for the limited number of terms of the series that I computed and hence I soon

used $\underline{u} = \underline{0.9}$, which point is also within the common region of convergence. The series converged much more rapidly when I used the latter value instead of the former.

The method of getting specific values for the expansions was this: In the developments above I substituted for $\underline{u}$, $\underline{0.9}$ and reduced my results, a process involving only algebraic work. Then each series gave me a value. This value is approximately the value of $\underline{v}$ when considered as a function of $\underline{u}$ in the neighborhood of both $\underline{u} = \underline{0}$ and each of the other critical points. The results which are correct to the second decimal place are given in the following table.

$u =$	I.	II.	III.	IV.
0	-0.401998	$+0.99393$	$-1.0285+0.4554\sqrt{-3}$	$-1.0285-0.4554\sqrt{-3}$
1	$+0.999711$	-0.40764	$-1.0262+0.4324\sqrt{-3}$	$-1.0262-0.4324\sqrt{-3}$
0	$-0.401998\,e^{\frac{3}{4}\pi i}$	$[-1.0285-0.4554\sqrt{-3}]\,e^{\frac{3}{4}\pi i}$	$+0.99393\,e^{\frac{3}{4}\pi i}$	$[-1.0285+0.4554\sqrt{-3}]\,e^{\frac{3}{4}\pi i}$
$e^{\frac{1}{4}\pi i}$	$+0.999711\,e^{\frac{3}{4}\pi i}$	$[-1.0262-0.4324\sqrt{-3}]\,e^{\frac{3}{4}\pi i}$	$-0.40764\,e^{\frac{3}{4}\pi i}$	$[-1.0262+0.4324\sqrt{-3}]\,e^{\frac{3}{4}\pi i}$
0	$-0.401998\,e^{\frac{3}{2}\pi i}$	$[-1.0285+0.4554\sqrt{-3}]\,e^{\frac{3}{2}\pi i}$	$[-1.0285-0.4553\sqrt{-3}]\,e^{\frac{3}{2}\pi i}$	$+0.99393\,e^{\frac{3}{2}\pi i}$
$e^{\frac{1}{2}\pi i}$	$+0.999711\,e^{\frac{3}{2}\pi i}$	$[-1.0262+0.4324\sqrt{-3}]\,e^{\frac{3}{2}\pi i}$	$[-1.0262-0.4324\sqrt{-3}]\,e^{\frac{3}{2}\pi i}$	$-0.40764\,e^{\frac{3}{2}\pi i}$
0	$-0.401998\,e^{\frac{1}{4}\pi i}$	$+0.99393\,e^{\frac{1}{4}\pi i}$	$[-1.0285+0.4553\sqrt{-3}]\,e^{\frac{1}{4}\pi i}$	$[-1.0285-0.4553\sqrt{-3}]\,e^{\frac{1}{4}\pi i}$
$e^{\frac{3}{4}\pi i}$	$+0.999711\,e^{\frac{1}{4}\pi i}$	$-0.40764\,e^{\frac{1}{4}\pi i}$	$[-1.0262+0.4324\sqrt{-3}]\,e^{\frac{1}{4}\pi i}$	$[-1.0262-0.4324\sqrt{-3}]\,e^{\frac{1}{4}\pi i}$
0	$-0.401998\,e^{\pi i}$	$[-1.0285-0.4553\sqrt{-3}]\,e^{\pi i}$	$+0.99393\,e^{\pi i}$	$[-1.0285+0.4553\sqrt{-3}]\,e^{\pi i}$
$e^{\pi i}$	$+0.999711\,e^{\pi i}$	$[-1.0262-0.4324\sqrt{-3}]\,e^{\pi i}$	$-0.40764\,e^{\pi i}$	$[-1.0262+0.4324\sqrt{-3}]\,e^{\pi i}$
0	$-0.401998\,e^{\frac{7}{4}\pi i}$	$[-1.0285+0.4553\sqrt{-3}]\,e^{\frac{7}{4}\pi i}$	$[-1.0285-0.4553\sqrt{-3}]\,e^{\frac{7}{4}\pi i}$	$+0.99393\,e^{\frac{7}{4}\pi i}$
$e^{\frac{5}{4}\pi i}$	$+0.999711\,e^{\frac{7}{4}\pi i}$	$[-1.0262+0.4324\sqrt{-3}]\,e^{\frac{7}{4}\pi i}$	$[-1.0262-0.4324\sqrt{-3}]\,e^{\frac{7}{4}\pi i}$	$-0.40764\,e^{\frac{7}{4}\pi i}$
0	$-0.401998\,e^{\frac{1}{2}\pi i}$	$+0.99393\,e^{\frac{1}{2}\pi i}$	$[-1.0285+0.4553\sqrt{-3}]\,e^{\frac{1}{2}\pi i}$	$[-1.0285-0.4553\sqrt{-3}]\,e^{\frac{1}{2}\pi i}$
$e^{\frac{3}{2}\pi i}$	$+0.999711\,e^{\frac{1}{2}\pi i}$	$-0.40764\,e^{\frac{1}{2}\pi i}$	$[-1.0262+0.4324\sqrt{-3}]\,e^{\frac{1}{2}\pi i}$	$[-1.0262-0.4324\sqrt{-3}]\,e^{\frac{1}{2}\pi i}$
0	$+0.401998\,e^{\frac{5}{4}\pi i}$	$[-1.0285-0.4553\sqrt{-3}]\,e^{\frac{5}{4}\pi i}$	$+0.99393\,e^{\frac{5}{4}\pi i}$	$[-1.0285+0.4553\sqrt{-3}]\,e^{\frac{5}{4}\pi i}$
$e^{\frac{7}{4}\pi i}$	$-0.999711\,e^{\frac{5}{4}\pi i}$	$[-1.0262-0.4324\sqrt{-3}]\,e^{\frac{5}{4}\pi i}$	$-0.40764\,e^{\frac{5}{4}\pi i}$	$[-1.0262+0.4324\sqrt{-3}]\,e^{\frac{5}{4}\pi i}$
0	-0.401998	$+0.99393$	$-1.0285+0.4553\sqrt{-3}$	$-1.0285-0.4553\sqrt{-3}$
∞	$+0.999711$	-0.40764	$-1.0262+0.4324\sqrt{-3}$	$-1.0262-0.4324\sqrt{-3}$

We shall now discuss the computed values as given in the table above. We have said that we may arbitrarily assume a connection at one point; and we choose that point as $\underline{v} = \underline{0}$. I have marked the sheets I., II., III. and IV; and I say that sheet I. shall be smooth. The table gives for the development of v as computed a certain value (0.401998). I further say that II. shall go over into III; III into IV. and IV. into II. Then by selecting the same values and following each through we can tell how the sheets are connected at every point, and which sheet is smooth. Let us consider as an example the smooth sheet. at $\underline{u} = \underline{0}$ I. is smooth; at $\underline{u} = \underline{1}$ II is smooth (0.40764); at $\underline{u} = \underline{e^{\frac{1}{4}\pi i}}$ III. is smooth, while at $\underline{u} = \underline{e^{\frac{1}{2}\pi i}}$ we see that IV. is smooth. At $u = e^{\frac{3}{4}\pi i}$ II. is again

smooth, and the permutation again begins. Did we then follow each computed value through we should have the permutations of the sheets complete. It will be noticed that $-\underline{0.401998}$ and $-\underline{0.40764}$ agree only to the second decimal place; perhaps I would have had my values agree to the fourth or fifth decimal place had I chosen $\underline{u} = \underline{0.95}$ instead of $\underline{u} = 0.\underline{9}$ for the point within the regions of convergence; but this is sufficiently close to show the way to the final goal - that of connecting up the sheets at all the points.

It is now proven, as stated before, that, having chosen my connection at one point, all the others are fixed by that one choice. I then have the following permutations, which I can read off from my table of computed values, and which

give me the order in which the sheets must be connected, the sheets as numbered in the parentheses being in each case the three that hang together at a branch point. The sheets outside the parentheses are then the smooth sheets.

Permutations.

$u = 0$ ------ I. (II, III., IV)

$u = 1$ ------ II. (I., III., IV)

$u = e^{\frac{1}{4}\pi i}$ ------ III. (I., IV., II.)

$u = e^{\frac{1}{2}\pi i}$ ------ IV. (I., II., III.)

$u = e^{\frac{3}{4}\pi i}$ ------ II. (I., III., IV.)

$u = e^{\pi i}$ ------ III. (I., IV., II.)

$u = e^{\frac{5}{4}\pi i}$ ------ IV. (I., II., III)

$u = e^{\frac{3}{2}\pi i}$ ------ II. (I., III., IV.)

$u = e^{\frac{7}{4}\pi i}$ ------ III. (I., IV., II.)

$u = \infty$ ------ I. (II., III., IV)

We shall now discuss some points connected with the manner in which the

sheets are connected.

We choose our junction lines running out to infinity from each of the branch points, as shown.

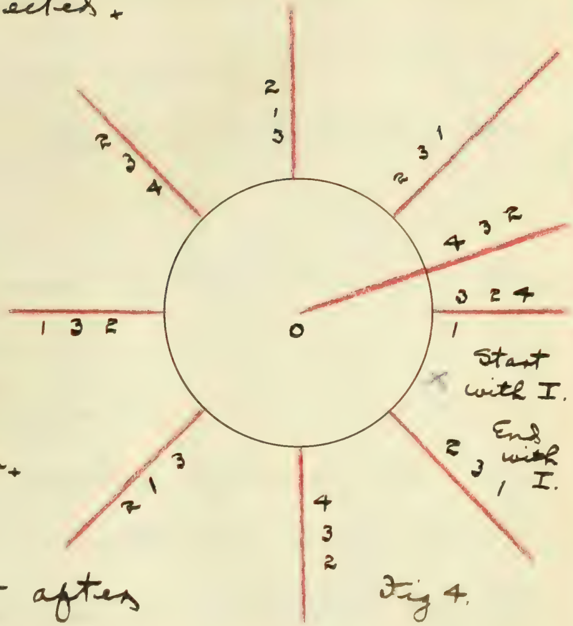


Fig 4.

The scheme will show that after we have gone around the circuit three times we again arrive at the starting-point. Let the starting point be sheet I at the cross in Fig 4, then as the junction lines are crossed one after another as we go in a positive direction the numbers will into what sheets we pass. Going around three times it is seen that we again arrive at sheet I when we come back

to the cross. This would show that the permutations as given are correct and hence the sheets are properly connected.

The discussion of the Riemann's Surface of the function:

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0$$

so far as the method of Cauchy is concerned is now complete as we have joined up the sheets at all the points where they hang together. However, a method much simpler and more elegant suggested itself in the process of the elaborate expansions, which method showed me that I could obtain the permutations by a consideration of the developments about $\underline{u = 0}$ only. This method consists in the simple transformations which amount to a successive rotation of the

$\underline{u}$ plane through an angle of $\underline{45^\circ}$, represented by $\underline{e^{\frac{1}{4}\pi i}}$.

Noticing the expansions about $\underline{u} = \underline{0}$ we see that as we go out from $\underline{v} = \underline{0}$ toward the point $\underline{v} = \underline{1}$ each one of the expressions gives a different path in which a branch starts outward from $\underline{0}$. If we can now get the image of the line from $\underline{v} = \underline{0}$ to $\underline{v} = \underline{1}$ in the $\underline{u}$ plane we shall then be able to get the images of all lines from $\underline{0}$ to one of the branch points by a simple transformation, which transformation will again give us the connection of the sheets.

We write here the first few terms of the expansions:

$$v_1 = -\frac{1}{2}u^3 - \frac{3}{32}u^{11} - \frac{3}{64}u^{19} - \frac{123}{4096}u^{27} - - - - -$$

$$v_2 = \sqrt[3]{2}\,u^{\frac{1}{3}} - \frac{1}{2}u^{\frac{9}{3}} + \frac{1}{4}\sqrt[3]{4}\,u^{\frac{17}{3}} - \frac{1}{6}u^{\frac{25}{3}} - - - - -$$

$$v_3 = \omega u^{\frac{1}{3}}.\sqrt[3]{2} - \frac{1}{2}u^{\frac{9}{3}} + \frac{1}{4}\sqrt[3]{4}\,\omega^2 u^{\frac{17}{3}} - - - - -$$

$$v_4 = \sqrt[3]{2}\,\omega^2 u^{\frac{1}{3}} - \frac{1}{2}u^{\frac{9}{3}} + \frac{1}{4}\sqrt[3]{4}\cdot\omega u^{\frac{17}{3}} - - - - -$$

Now it is seen that when $\underline{u}$ is small the higher powers drop out in comparison with the lower; and for very small values of $\underline{u}$ the first terms will show the directions in which the branches of the image start out from $\underline{v} = \underline{0}$. These terms then show us that as $\underline{u}$ goes in a positive direction the first branch gives a negative value to $\underline{v}_1$; and as the value is real when $\underline{u}$ is real the path is to the left along the axis of reals. Again, $\underline{v}_2$ is positive when $\underline{u}$ is positive; hence another branch goes to the right along the axis of reals. This is sheet II., one of the three that are connected. Similarly the factors $\underline{\omega}$ and $\underline{\omega}^2$ show that the other two paths are at angles of 120° and 240° with the axis of reals respectively. The following diagram will show the paths:

We shall adopt the method of coloring the portion of the image in each sheet always with the same color, the colors for the different sheets being as shown in Fig 5.

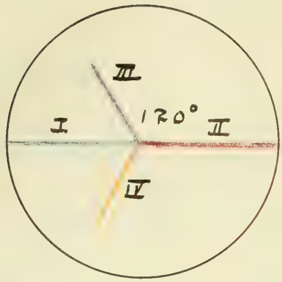


Fig 5.

We have already plotted the paths of two branches of the image - I and II - as u increases from 0 to 1. The paths for III. and IV. I obtained by plotting for varying values of u. This work showed me that the paths went outside of the unit circle; the two were symmetrical with respect to the axis of reals, and again met at an angle of 120°. The whole image is then as shown in Fig. 6.

As we go around 0 we see that, since I. is smooth, II. goes over into III, and III into IV.

From Fig. 6 we also see that at the point $u = 1$ II is smooth while I. goes into III. and III into IV. This we see from the order in which we cross the branches of the image when going around $\underline{u} = \underline{1}$.

Fig 6.

We must now get the image of the line joining the branch point $\underline{0}$ with the branch point $e^{\frac{1}{4}\pi i}$. As we have stated before we shall make the transformation $\bar{\underline{u}} = \underline{u e^{\frac{1}{4}\pi i}}$ to get the image of the line mentioned. This simply rotates our $\underline{u}$ plane through an angle of $\underline{45}°$; and we shall find that, since the angles in the $\underline{v}$ plane are trebled, the rotation of that plane is three times as great, or through an angle of 135°. I give

the work complete for the case of the critical point $u = e^{\frac{1}{4}\pi i}$.

For $u = e^{\frac{1}{4}\pi i}$.

Make transformation $\bar{u} = u\, e^{\frac{1}{4}\pi i}$.

Designating the sheets for this critical point by $v_1^{I}, v_2^{I}, v_3^{I}, v_4^{I}$ we have:

$$v_1^{I} = -\frac{1}{2}\bar{u}^{3} - \frac{3}{32}\bar{u}^{11} - \frac{3}{64}\bar{u}^{19} - \frac{123}{4096}\bar{u}^{27} \text{------}$$

$$= e^{\frac{3}{4}\pi i}\left(-\frac{1}{2}u^{3} - \frac{3}{32}u^{11} - \frac{3}{64}u^{19} \text{----}\right) = e^{\frac{3}{4}\pi i} v_1.$$

$$v_2^{I} = \sqrt[3]{2}\,\bar{u}^{\frac{1}{3}} - \frac{1}{2}\bar{u}^{\frac{9}{3}} + \frac{1}{4}\cdot\sqrt[3]{4}\,\bar{u}^{\frac{17}{3}} \text{------}$$

$$= \sqrt[3]{2}\,u^{\frac{1}{3}} e^{\frac{1}{12}\pi i} - \frac{1}{2}u^{\frac{9}{3}} e^{\frac{9}{12}\pi i} + \frac{1}{4}\cdot\sqrt[3]{4}\,u^{\frac{17}{3}} e^{\frac{17}{12}\pi i} \text{---}$$

$$= \sqrt[3]{2}\,u^{\frac{1}{3}}\, e^{\frac{3}{4}\pi i}\, e^{-\frac{8}{12}\pi i} - \frac{1}{2}u^{\frac{9}{3}} e^{\frac{3}{4}\pi i} + \frac{1}{4}\cdot\sqrt[3]{4}\,u^{\frac{17}{3}} e^{\frac{3}{4}\pi i}\, e^{\frac{8}{12}\pi i} \text{----}$$

$$= e^{\frac{3}{4}\pi i}\left(\sqrt[3]{2}\,\omega^{2}.\,u^{\frac{1}{3}} - \frac{1}{2}u^{\frac{9}{3}} + \frac{1}{4}\cdot\sqrt[3]{4}.\,\omega.\,u^{\frac{17}{3}} \text{-----}\right.$$

$$= e^{\frac{3}{4}\pi i} v_4$$

$$v_3^{I} = \sqrt[3]{2}.\,\omega.\,u^{\frac{1}{3}} - \frac{1}{2}\bar{u}^{\frac{9}{3}} + \frac{1}{4}\cdot\sqrt[3]{4}\,\bar{u}^{\frac{17}{3}}\,\omega^{2} \text{------}$$

$$= \sqrt[3]{2}\,e^{\frac{1}{12}\pi i}\,\omega.\,u^{\frac{1}{3}} - \frac{1}{2}e^{\frac{9}{12}\pi i}\,u^{\frac{9}{3}} + \frac{1}{4}\cdot\sqrt[3]{4}.\,\omega^{2}.\,e^{\frac{17}{12}\pi i}\,u^{\frac{17}{3}} \text{---}$$

$$= \sqrt[3]{2}\,e^{\frac{9}{12}\pi i}\,u^{\frac{1}{3}} - \frac{1}{2}e^{\frac{9}{12}\pi i}\,u^{\frac{9}{3}} + \frac{1}{4}\cdot\sqrt[3]{4}.\,e^{\frac{33}{12}\pi i}\,u^{\frac{17}{3}} \text{---}$$

$$= e^{\frac{3}{4}\pi i}\left(\sqrt[3]{2}\,u^{\frac{1}{3}} - \frac{1}{2}u^{\frac{9}{3}} + \frac{1}{4}\cdot\sqrt[3]{4}.\,u^{\frac{17}{3}} \text{-----}\right)$$

$$= e^{\frac{3}{4}\pi i} v_2.$$

$$v_4^{I} = \sqrt[3]{2}\,\omega^2 \bar{u}^{\frac{1}{3}} - \frac{1}{2}\bar{u}^{\frac{9}{3}} + \frac{1}{4}\cdot\sqrt[3]{4}\,\bar{u}^{\frac{17}{3}}\omega \cdots\cdots$$

$$= \sqrt[3]{2}\,\omega^2 e^{\frac{1}{12}\pi i} u^{\frac{1}{3}} - \frac{1}{2}e^{\frac{9}{12}\pi i} u^{\frac{9}{3}} + \frac{1}{4}\cdot\sqrt[3]{4}\,\omega\, e^{\frac{17}{12}\pi i} u^{\frac{17}{3}} \cdots$$

$$= e^{\frac{3}{4}\pi i}\left(\sqrt[3]{2}\cdot\omega\, u^{\frac{1}{3}} - \frac{1}{2}u^{\frac{9}{3}} + \frac{1}{4}\cdot\sqrt[3]{4}\,u^{\frac{17}{3}}\omega^2 \cdots\cdots\right.$$

$$= e^{\frac{3}{4}\pi i} v_3 .$$

Collecting results I then have:

$$v_1^{I} = e^{\frac{3}{4}\pi i} v_1$$
$$v_2^{I} = e^{\frac{3}{4}\pi i} v_4$$
$$v_3^{I} = e^{\frac{3}{4}\pi i} v_2$$
$$v_4^{I} = e^{\frac{3}{4}\pi i} v_3$$

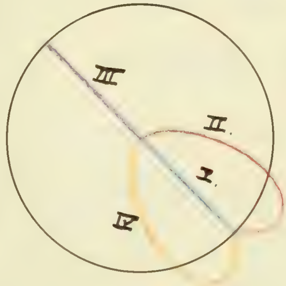


Fig 7.

The above shows us that the image of the line connecting the branch points is rotated in a positive direction through an angle of $\underline{135}°$, as given by $e^{\frac{3}{4}\pi i}$. That the image will be the same is shown by the fact that v_1, v_2, v_3 and v_4 are found here, the factor $e^{\frac{3}{4}\pi i}$ being the only new factor introduced by the transformation.

We may now apply this method to any of the expansions about the critical points. For the point $\underline{u = e^{\frac{1}{2}\pi i}}$ we then make the transformation:

$\bar{u} = e^{\frac{1}{2}\pi i} u$, and by reductions similar to those above we obtain:

$$v_1^{II} = e^{\frac{3}{2}\pi i} v_1 = e^{\frac{3}{4}\pi i} v_1^{I}$$
$$v_2^{II} = e^{\frac{3}{2}\pi i} v_3 = e^{\frac{3}{4}\pi i} v_4^{I}$$
$$v_3^{II} = e^{\frac{3}{2}\pi i} v_4 = e^{\frac{3}{4}\pi i} v_2^{I}$$
$$v_4^{II} = e^{\frac{3}{2}\pi i} v_2 = e^{\frac{3}{4}\pi i} v_3^{I}$$

This shows us that our image is rotated 135° further.

The figures (7 and 8) show us that at $u = e^{\frac{1}{4}\pi i}$ sheet III is smooth, while I. goes over into IV and IV. into II. Likewise

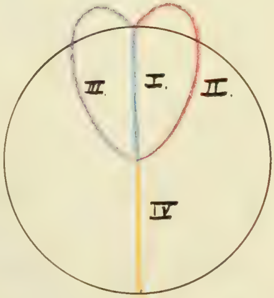


Fig. 8

at $u = e^{\frac{1}{2}\pi i}$ IV. is smooth while I. goes over into II. and II. into III. I preserve the color of each sheet.

For $u = e^{\frac{3}{4}\pi i}$

Make transformation: $\bar{u} = u\, e^{\frac{3}{4}\pi i}$

We have as results of our transformation:

$v_1^{III} = e^{\frac{1}{4}\pi i} v_1 = e^{\frac{3}{4}\pi i} v_1^{II}$

$v_2^{III} = e^{\frac{1}{4}\pi i} v_2 = e^{\frac{3}{4}\pi i} v_4^{II}$

$v_3^{III} = e^{\frac{1}{4}\pi i} v_3 = e^{\frac{3}{4}\pi i} v_2^{II}$

$v_4^{III} = e^{\frac{1}{4}\pi i} v_4 = e^{\frac{3}{4}\pi i} v_3^{II}$

again the image is rotated 135° further.

From the figure we see that here II is smooth; I goes over into III and III into IV.

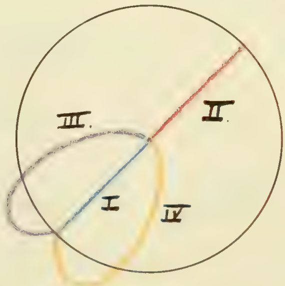


Fig. 9.

The fact that $v_1^{III} = e^{\frac{1}{4}\pi i} v_1$ shows that our original figure is rotated through an angle of 45° from its first position and 135° from its next preceding position.

The point $u = e^{\pi i} = -1$ we treat as $u = e^{\pi i}$ as this is just as simple as to make $\underline{u} = \underline{-1}$.

For $u = e^{\pi i}$.

Make transformation: $\bar{u} = u - e^{\pi i}$

$$\therefore \quad v_1^{IV} = e^{\pi i} v_1 = e^{\frac{3}{4}\pi i} v_1^{III}$$
$$v_2^{IV} = e^{\pi i} v_4 = e^{\frac{3}{4}\pi i} v_4^{III}$$
$$v_3^{IV} = e^{\pi i} v_2 = e^{\frac{3}{4}\pi i} v_2^{III}$$
$$v_4^{IV} = e^{\pi i} v_3 = e^{\frac{3}{4}\pi i} v_3^{III}$$

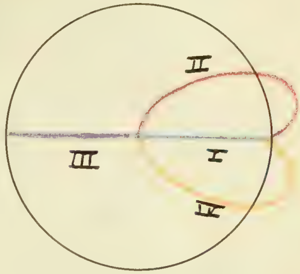


Fig. 10.

The connection of the sheets is shown by the figure. Sheet III. is smooth.

For $u = e^{\frac{5}{4}\pi i}$

Make transformation: $\bar{u} = u e^{\frac{5}{4}\pi i}$.

$$\therefore \quad v_1^{I} = e^{\frac{7}{4}\pi i} v_1 = e^{\frac{3}{4}\pi i} v_1^{IV}$$
$$v_2^{I} = e^{\frac{7}{4}\pi i} v_3 = e^{\frac{3}{4}\pi i} v_4^{IV}$$
$$v_3^{I} = e^{\frac{7}{4}\pi i} v_4 = e^{\frac{3}{4}\pi i} v_2^{III}$$
$$v_4^{I} = e^{\frac{7}{4}\pi i} v_2 = e^{\frac{3}{4}\pi i} v_3^{IV}$$

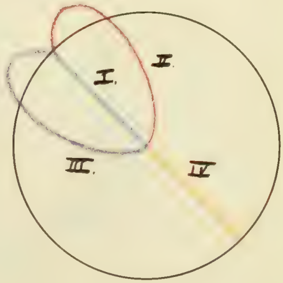


Fig 11.

At this point we have the following permutation showing connections: IV (I., II., III).

For $u = e^{\frac{3}{2}\pi i}$

Make transformation $\bar{u} = u e^{\frac{3}{2}\pi i}$.

$$\therefore \quad v_1^{VI} = e^{\frac{1}{2}\pi i} v_1 = e^{\frac{3}{4}\pi i} v_1^{V}$$

$$v_2^{VI} = e^{\frac{1}{2}\pi i} v_2 = e^{\frac{3}{4}\pi i} v_4^{V}$$

$$v_3^{VI} = e^{\frac{1}{2}\pi i} v_3 = e^{\frac{3}{4}\pi i} v_2^{V}$$

$$v_4^{VI} = e^{\frac{1}{2}\pi i} v_4 = e^{\frac{3}{4}\pi i} v_3^{V}$$

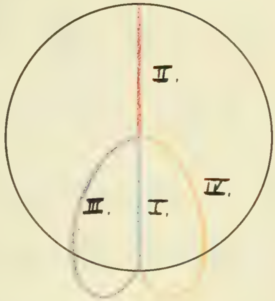


Fig. 12.

The following permutation shows the connection at this point: II. (I., III., IV.)

For $u = e^{\frac{7}{4}\pi i}$

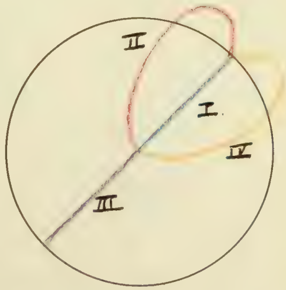
Make transformation: $\bar{u} = u e^{\frac{7}{4}\pi i}$.

$$v_1^{VIII} = e^{\frac{5}{4}\pi i} v_1 = e^{\frac{3}{4}\pi i} v_1^{VI}$$

$$v_2^{VIII} = e^{\frac{5}{4}\pi i} v_4 = e^{\frac{3}{4}\pi i} v_4^{VI}$$

$$v_3^{VIII} = e^{\frac{5}{4}\pi i} v_2 = e^{\frac{3}{4}\pi i} v_2^{VI}$$

$$v_4^{VIII} = e^{\frac{5}{4}\pi i} v_3 = e^{\frac{3}{4}\pi i} v_3^{VI}$$



The connection of the sheets here is given by the permutation: III. (I., IV., II.)

Fig. 13

For $u = \infty$.

We have shown how this critical point is treated. We make the transformation

$\bar{u} = \frac{1}{u} \therefore u = \frac{1}{\bar{u}}$ in the function

$u^4 - v^4 + 2uv(1 - u^2v^2) = 0$; and the result comes out identically the same as the original function. Hence the paths meet at infinity at the same angles at which they meet at the point $u = 0$. Fig. 14 shows these angles.

The permutation is:

I. (II., III., IV.).

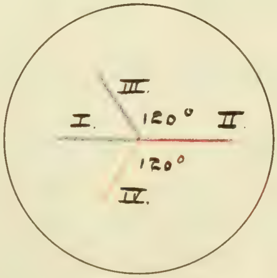


Fig. 14

We have now obtained the connection of the sheets of our Riemann's Surface in two different ways; and both give us identical results. We may then regard our problem as solved.

We shall now proceed to the study of the Riemann's Surface of the second of our modular functions. We have given the function:

$$(1)\quad u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$$

We again form the partial derivative of the function with respect to $\underline{u}$.

$$(2)\quad \frac{\partial F(u,v)}{\partial u} = 6u^5 + 20u^3v - 10uv^4 + 4v - 20u^4v^5 = 0.$$

Now eliminate $\underline{u}$ between (1) and (2) by forming the resultant in $\underline{v}$. The resultant in determinant form is the following:

$$\begin{vmatrix}
1 & -4v^5 & 5v^2 & 0 & -5v^4 & 4v & -v^6 & 0 & 0 & 0 & 0 \\
0 & 1 & -4v^5 & 5v^2 & 0 & -5v^4 & 4v & -v^6 & 0 & 0 & 0 \\
0 & 0 & 1 & -4v^5 & 5v^2 & 0 & -5v^4 & 4v & -v^6 & 0 & 0 \\
0 & 0 & 0 & 1 & -4v^5 & 5v^2 & 0 & -5v^4 & 4v & -v^6 & 0 \\
0 & 0 & 0 & 0 & 1 & -4v^5 & 5v^2 & 0 & -5v^4 & 4v & -v^6 \\
6 & -20v^5 & 20v^2 & 0 & -10v^4 & 4v & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & -20v^5 & 20v^2 & 0 & -10v^4 & 4v & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & -20v^5 & 20v^2 & 0 & -10v^4 & 4v & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & -20v^5 & 20v^2 & 0 & -10v^4 & 4v & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & -20v^5 & 20v^2 & 0 & -10v^4 & 4v & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & -20v^5 & 20v^2 & 0 & -10v^4 & 4v
\end{vmatrix} = 0$$

when this determinant is reduced and solved for v we obtain:

$$v^8 = 1 \quad \therefore v = \sqrt[8]{1}$$

Substituting the eight values for v in the equation:

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$$

we obtain the critical points and have the following table:

Critical Pt.	Order	Branch Pt.	Times	Ordinary Pt.
$u = 1$	4	-1	5	1
$e^{\frac{1}{4}\pi i}$	4	$-e^{\frac{3}{4}\pi i}$	5	$e^{\frac{3}{4}\pi i}$
$e^{\frac{1}{2}\pi i}$	4	$e^{\frac{1}{2}\pi i}$	5	$-e^{\frac{1}{2}\pi i}$
$e^{\frac{3}{4}\pi i}$	4	$-e^{\frac{1}{4}\pi i}$	5	$e^{\frac{1}{4}\pi i}$
$e^{\pi i}$	4	1	5	-1
$e^{\frac{5}{4}\pi i}$	4	$e^{\frac{3}{4}\pi i}$	5	$-e^{\frac{3}{4}\pi i}$
$e^{\frac{3}{2}\pi i}$	4	$-e^{\frac{1}{2}\pi i}$	5	$e^{\frac{1}{2}\pi i}$
$e^{\frac{7}{4}\pi i}$	4	$e^{\frac{1}{4}\pi i}$	5	$-e^{\frac{1}{4}\pi i}$
0	4	0	5	0
∞	4	∞	5	∞

This table differs only in the order of the critical points and the number of times

each branch point occurs for its correspond-ing critical point. Since the order of the critical points is 4 then of the six sheets of the Riemann's surface five hang together at the branch points while one is smooth at each point. We must then again determine the connection at every point; and in order to do so we again obtain our developments for the six branches of the function at every point in a manner strictly analogous to that in which we obtained the developments for the various branches in the function first discussed. The algebraic work offers no added dif-ficulty except the extracting of the fifth root of a series, which, though labori-ous, is possible.

We have then as the developments about the point $u=0$ the following:

$$v_1 = -\frac{1}{4}u^5 - \frac{5}{64}u^{13} - \frac{45}{1024}u^{21} - \frac{245}{8192}u^{29} - \frac{2935}{131072}u^{37} - - - - -$$

$$v_2 = \sqrt[5]{4}\,u^{\frac{1}{5}} - \frac{1}{2}\sqrt[5]{8}\,u^{\frac{9}{5}} + \frac{1}{2}\sqrt[5]{16}\,u^{\frac{17}{5}} - \frac{3}{4}u^5 + \frac{1}{2}\sqrt[5]{2}\,u^{\frac{33}{5}} - - - - -$$

$$v_3 = \sqrt[5]{4}.e^{\frac{2}{5}\pi i}u^{\frac{1}{5}} - \frac{1}{2}\sqrt[5]{8}.e^{\frac{8}{5}\pi i}u^{\frac{9}{5}} + \frac{1}{2}\sqrt[5]{16}\,e^{\frac{4}{5}\pi i}u^{\frac{17}{5}} - \frac{3}{4}u^5 + \frac{1}{2}\sqrt[5]{2}\,e^{\frac{6}{5}\pi i}u^{\frac{33}{5}} \cdots$$

$$v_4 = \sqrt[5]{4}.\,e^{\frac{4}{5}\pi i}u^{\frac{1}{5}} - \frac{1}{2}\sqrt[5]{8}.e^{\frac{6}{5}\pi i}u^{\frac{9}{5}} + \frac{1}{2}\sqrt[5]{16}.e^{\frac{8}{5}\pi i}u^{\frac{17}{5}} - \frac{3}{4}u^5 + \frac{1}{2}\sqrt[5]{2}.e^{\frac{2}{5}\pi i}u^{\frac{33}{5}} \cdots$$

$$v_5 = \sqrt[5]{4}\,e^{\frac{6}{5}\pi i}u^{\frac{1}{5}} - \frac{1}{2}\sqrt[5]{8}\,e^{\frac{4}{5}\pi i}u^{\frac{9}{5}} + \frac{1}{2}\sqrt[5]{16}.\,e^{\frac{2}{5}\pi i}u^{\frac{17}{5}} - \frac{3}{4}u^5 + \frac{1}{2}\sqrt[5]{2}.\,e^{\frac{8}{5}\pi i}u^{\frac{33}{5}} \cdots$$

$$v_6 = \sqrt[5]{4}\,e^{\frac{8}{5}\pi i}u^{\frac{1}{5}} - \frac{1}{2}\sqrt[5]{8}\,e^{\frac{2}{5}\pi i}u^{\frac{9}{5}} + \frac{1}{2}\sqrt[5]{16}\,e^{\frac{6}{5}\pi i}u^{\frac{17}{5}} - \frac{3}{4}u^5 + \frac{1}{2}\sqrt[5]{2}.\,e^{\frac{4}{5}\pi i}u^{\frac{33}{5}} \cdots$$

As we found in our previous discussion that we could by transformations obtain the permutations of the sheets at the different branch points so we employ that method only in this case. The transformations here are analogous to those in the preceding case.

We must first, however, find the image in the $\underline{u}$ plane of the line connecting the two branch points $\underline{v} = \underline{0}$ and $\underline{v} = \underline{1}$. To do this we shall proceed as before — substitute

various values of u between 0 and 1 and thus obtain from the developments points on the image. Since the coefficients in the five series which represent the values of the function in the sheets which hang together are the same numerically I give below a table involving these and the varying values of u chosen in order to give me the graph of the image.

It might be said here that the common region of convergence for the developments about any two points is the same as in the preceding case - up to the next critical point.

	$u = 0.2$	0.4	0.6	0.8	0.9
$\sqrt[5]{4}\, u^{\frac{1}{5}}$	0.956	1.098	1.191	1.262	1.292
$\frac{1}{2}\sqrt[5]{8}\, u^{\frac{9}{5}}$	0.042	0.146	0.302	0.507	0.627
$\frac{1}{2}\sqrt[5]{16}\, u^{\frac{17}{5}}$	0.004	0.039	0.153	0.408	0.608
$\frac{3}{4} u^5$	0.0002	0.008	0.058	0.246	0.443
$\frac{1}{2}\sqrt[5]{2}\, u^{\frac{33}{5}}$	0.00001	0.001	0.0197	0.132	0.287

By substituting the above numerical values for the coefficients multiplied by the particular values of u and multiplying each by its proper power of e we have a set of five complex quantities in each case. We must then by vector addition obtain the various points on the different branches of the function, and enough of these to give us the graph of the image.

First, however, let us investigate the developments in order to find the angles at which the branches of the image proceed outward from the origin. For very small values of u the higher powers drop out in comparison with the first term; hence the latter may be used to show angles at which the branches start. For v_1, or the first branch we see that

when $\underline{v}$ is negative $\underline{u}$ is positive and vice versa. As $\underline{u}$ goes from $\underline{0}$ to $\underline{1}$, $\underline{v}_1$ goes from $\underline{0}$ to $-\underline{1}$ and along the axis of reals. The branch represented by $\underline{v}_2$ is also real for real values of $\underline{u}$ and positive; hence the image of the second branch is on the axis of reals and positive. The factors $e^{\frac{2}{5}\pi i}, e^{\frac{4}{5}\pi i}, e^{\frac{6}{5}\pi i}, e^{\frac{8}{5}\pi i}$ show us that the other four branches proceed outward from the origin at angles of $72°$, $144°$, $216°$ and $288°$ for v_3, v_4, v_5 and v_6 respectively. We have then the following figure which shows the direction of the image of the branches in the vicinity of the origin.

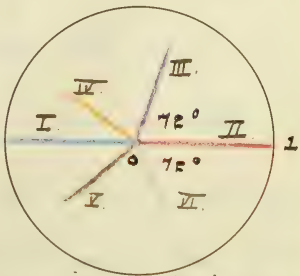


Fig. 15.

We have here the angles at which the branches <u>start</u> but that only. By considering the equation:

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$$

we see that as <u>u</u> varies between <u>0</u> and <u>1</u> we have:

$$v^6 + av^5 + bv^4 + cv^2 - dv - e = 0$$

This equation has six roots which represent the branches and hence start out at $\underline{u} = \underline{1}$ with the same angles as those at which the branches proceed outward from the origin. But we must have the exact paths of the branches for v_3, v_4, v_5 and $\underline{v_6}$ between the origin and the point <u>1</u>. We know that I. and II. (corresponding to $\underline{v_1}$ and $\underline{v_2}$) will remain on the axis of reals, and we may infer from the results of the preceding case that the four branches to be determined will be placed

symmetrically with respect to the axis of reals. We then use our table of computed values and by the addition of vectors obtain various points on the four paths.

Let us take the third branch, represented by: $v_3 = \sqrt[5]{4}\, e^{\frac{2}{5}\pi i} u^{\frac{1}{5}} - \frac{1}{2}\sqrt[5]{8}\, e^{\frac{8}{5}\pi i} u^{\frac{9}{5}} + \frac{1}{2}\sqrt[5]{16}\, e^{\frac{4}{5}\pi i} u^{\frac{17}{5}}$

$$- \frac{3}{4} u^5 + \frac{1}{2}\sqrt[5]{2}\, e^{\frac{6}{5}\pi i} u^{\frac{33}{5}} - - - - - -$$

and compute two characteristic cases, the first where $\underline{u}$ is in the neighborhood of $\underline{0}$ and the second where $\underline{u}$ is in the neighborhood of the value $\underline{1}$.

① $= 0.956\, e^{\frac{2}{5}\pi i}$
② $= -0.0418\, e^{\frac{8}{5}\pi i}$
③ $=$ ① $+$ ②
From vector addition.
The remaining terms of the series are so small in comparison with the first two that

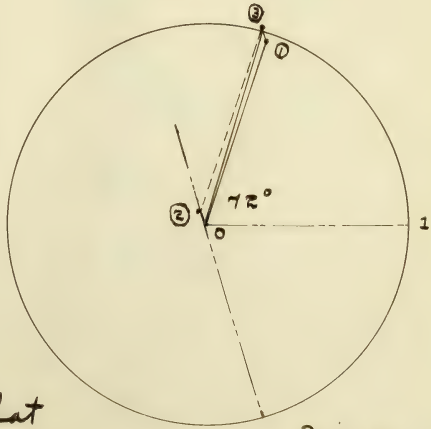


Fig. 16

we may consider ③ as the point on the branch III. for $\underline{u} = \underline{0.2}$. Evidently the curve is asymptotic to the straight line making an angle of 72° with the axis of reals. The extremity of VI. is likewise asymptotic to the line making an angle of −72° with the axis of reals. We now give the determination of the point on the graph of III for the value of $\underline{u} = \underline{0.9}$.

① $= 1.292\, e^{\frac{2}{5}\pi i}$

② $= -0.627\, e^{\frac{8}{5}\pi i}$

③ = ① + ②

④ $= 0.608\, e^{\frac{4}{5}\pi i}$

⑤ = ③ + ④

⑥ $= -0.443\, u^5$

⑦ = ⑤ + ⑥

⑧ $= 0.286\, e^{\frac{6}{5}\pi i}$

⑨ = ⑦ + ⑧.

The two points plotted are characteristic of all.

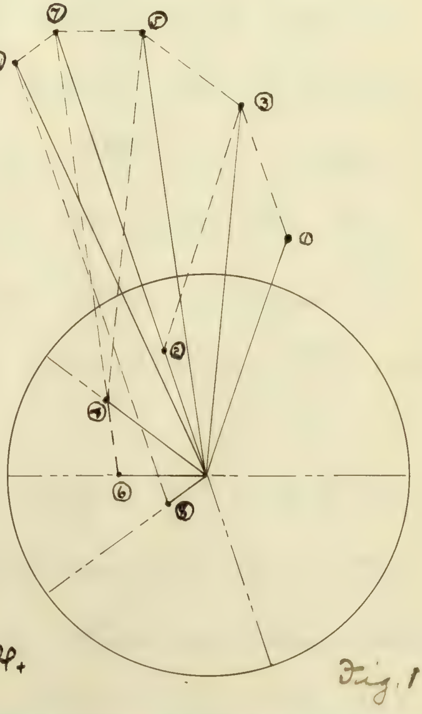


Fig. 17

It is seen that in plotting for $u = 0.9$ the series are not expanded far enough to give the point on the graph accurately; but it is also seen that enough terms have been taken to show about where the point on the graph will lie. It will be a little further to the left and nearer the point -1. The points for 0.4, 0.6 and 0.8 lie between those for 0.2 and 0.9 and all are outside of the unit circle. The left end of the branch is asymptotic to a line making an angle of 144° with the axis of reals. Branch VI is symmetrical to branch III., the axis of reals being the axis of symmetry.

When we plot branches IV. and V. we find them within the branches III and VI, and this happens because there are some vectors

in their expansions which must be subtracted and which throw the point further within the unit circle. In fact IV. and I. never leave the unit circle. They are further symmetrically placed with respect to the axis of reals and IV makes angles of 144° and 36° with the axis of reals at the points of departure. We have then sufficient data to draw the graph of the image of the function, and this graph is the following:

In discussing this graph further the color scheme adopted for the sheets holds throughout.

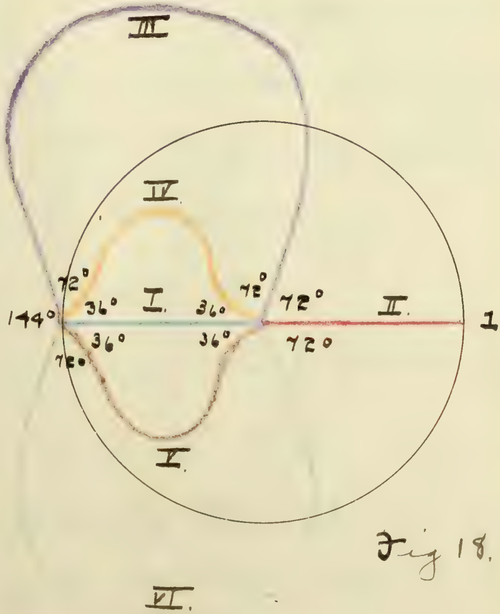


Fig 18.

It remains to show that we may now by suitable transformations rotate the u plane through an angle which will bring the axis of reals to such a position that it will pass through the next critical point. Then our first transformation, as in our previous discussion is:

$$\bar{u} = u e^{\frac{1}{4}\pi i}.$$

The effect of this is to rotate the axis through an angle of 45° in a positive direction. When we make this transformation in our developments for v we find that instead of the permutation:

$$\mathrm{I}.(\mathrm{II}., \mathrm{III}., \mathrm{IV}, \mathrm{V}, \mathrm{VI})$$

we shall have another, which proves to be:

$$\mathrm{II}\ (\mathrm{I}, \mathrm{IV}, \mathrm{III}, \mathrm{VI}, \mathrm{V}).$$

Let us now consider through how great an angle the image is rotated because of

the transformation in u.

The simple expansion:

$$v_1^{I} = -\frac{1}{4}\bar{u}^5 - \frac{5}{64}\bar{u}^{13} - \frac{45}{1024}u^{21} - \frac{245}{8192}u^{29} - \frac{2935}{131072}u^{37}\cdots$$

where $\bar{u} = u\,e^{\frac{1}{4}\pi i}$ becomes:

$$= e^{\frac{5}{4}\pi i}\left(-\frac{1}{4}u^5 - \frac{5}{64}u^{13} - \frac{45}{1024}u^{21} - \frac{245}{8192}u^{29}\cdots\right)$$

and shows us that while we rotate the u plane through an angle of $45°$ the image is rotated through an angle of $225°$, as shown by the factor $e^{\frac{5}{4}\pi i}$. The image remains the same in contour, but the branches change places, the new permutation being given above.

We now make successively the transformations:

$$\bar{u} = u\,e^{\frac{1}{2}\pi i}$$
$$\bar{u} = u\,e^{\frac{3}{4}\pi i}$$
$$\bar{u} = u\,e^{\pi i}$$
$$\bar{u} = u\,e^{\frac{5}{4}\pi i}$$
$$\bar{u} = u\,e^{\frac{3}{2}\pi i}$$
$$\bar{u} = u\,e^{\frac{7}{4}\pi i}.$$

Each of these brings in interchanges of series analogous to those in the preceding case. These interchanges are given by the permutations which are given below.

Now, as each transformation rotates the image in a positive direction, and through an angle of $\underline{225}°$ then after we have made eight rotations we arrive at the point from which we started. Finally for $u = \infty$, by the transformation:

$$\bar{u} = \frac{1}{u} \quad \therefore \quad u = \frac{1}{\bar{u}}$$ we get the same expression as the original function.

Eight rotations of $\underline{225}°$ each amount to a rotation through $1\underline{800}°$ or $\underline{5 \times 360}°$, showing that after eight transformations we have gone around the circuit five times and we are, as we should be, at the starting point.

We give the permutations which show how the sheets go over into each other:

at: $u = 0$ $\quad$ I. (II, III. IV. V. VI.)

$u = 1$ $\quad$ II. (I. IV. III. VI. V.)

$u = e^{\frac{1}{4}\pi i}$ $\quad$ IV. (I. VI. V. III. II)

$u = e^{\frac{1}{2}\pi i}$ $\quad$ VI. (I. III. II. V. IV.)

$u = e^{\frac{3}{4}\pi i}$ $\quad$ III. (I. V. IV. II. VI.)

$u = e^{\pi i}$ $\quad$ V. (I. II. VI. IV. III.)

$u = e^{\frac{5}{4}\pi i}$ $\quad$ II. (I. IV. III. VI. V.)

$u = e^{\frac{3}{2}\pi i}$ $\quad$ IV. (I. VI. V. III. II.)

$u = e^{\frac{7}{4}\pi i}$ $\quad$ VI. (I. III. II. V. IV.)

$u = \infty$ $\quad$ III (I. V. IV. II. VI.)

We have then the data for connecting up the sheets of the Riemann's Surface of the function under consideration; and as this is the end sought we may then consider the problem of the Riemann's Surface of this second modular function as solved.

To present to the eye more clearly the exact nature of the connection of the sheets of the Riemann's Surfaces of the two functions considered in this thesis I have constructed models on the conventional plan, which I have added to the collection of mathematical models in possession of the University. I also have placed in the body of this work a photograph of each of the models, which photographs show partly what the models are intended to illustrate: The Riemann's Surfaces of the Modular Functions:

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0.$$

and $$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0.$$

End.